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The action–angle Wigner function: a discrete, finite and algebraic phase space formalism

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Abstract. The action–angle representation in quantum mechanics is conceptually quite different from its classical counterpart and motivates a *canonical discretization* of the phase space. In this work, a discrete and finite-dimensional phase space formalism, in which the phase space variables are discrete and the time is continuous, is developed and the fundamental properties of the discrete Weyl–Wigner–Moyal quantization are derived. The action–angle Wigner function is shown to exist in the semi-discrete limit of this quantization scheme. A comparison with other formalisms which are not explicitly based on canonical discretization is made. Fundamental properties that an action–angle phase space distribution respects are derived. The dynamical properties of the action–angle Wigner function are analysed for discrete and finite-dimensional model Hamiltonians. The limit of the discrete and finite-dimensional formalism including a discrete analogue of the Gaussian wavefunction spread, viz. the binomial wavepacket, is examined and shown by examples that standard (continuum) quantum mechanical results can be obtained as the dimension of the discrete phase space is extended to infinity.

1. Introduction and motivation

The continuous Wigner function formalism [1–4] is a crucial element for the standard quantum phase space not only as a calculational tool but also as a powerful conceptual bridge between classical and quantum mechanics. We now know a number of phase space distribution functions other than Wigner, of which a small number of them are [5], the Q-distribution of Husimi, the P-distribution of Glauber and Sudarshan as well as Drummond, Gardiner and Walls which are powerful tools in phase space quantum optics. Although, the majority of the formulations of generalized phase space distribution functions were based on the standard q, p representation, those based on other canonical phase space observables were also devised. In particular, those distributions represented in terms of action and angle (AA) observables were also developed more recently [6, 7] and applied to a limited number of simple bound state problems [8].

The classical action–angle phase space approach is based on constructing (provided they exist) a sufficient number of independent functions of the phase space observables $J(p, q; t)$, which are constants of time on some classical Hamiltonian manifold. By an old theorem proved by Liouville [9], a canonical transformation (CT) can then be found $(q, p) \overset{CT}{\Longrightarrow} (J, \theta)$ from the old coordinates (q, p) to a new set, namely the action–angle $(J = J(p, q; t), \theta = \theta(p, q; t))$ coordinates, where the action coordinates are given by those independent functions of phase space observables and the angle ones are their canonical partners. It was proved by Liouville that if as many action coordinates can be constructed as the system's independent degrees of freedom, then the equations of motion representing the dynamical system are completely integrable [10]. In most general circumstances, the action variables are real valued and the angle ones are defined on the 2π interval.

However, in quantum mechanics one expects a completely different picture [6] played by the canonical action–angle pair due to the fact that the action observable is only allowed to take values on the set of integers $\mathbb{Z}$ in units of the Planck constant $\hbar$. Contrary to the classical case, the AA Wigner function representation of quantum systems has proven to be a challenge. Here, the quantum harmonic oscillator (QHO) has a central importance as being the simplest problem in the standard $\hat{p}$, $\hat{q}$ representation but is notoriously difficult in the action–angle one due to the additional fact that the eigenvalues of its action operator is expected to span only the non-negative integers [8, 11–13].

The search for the quantum mechanical operator counterpart of the classical action–angle observables is also known as the quantum phase problem which has been a long-standing issue of quantum mechanics since Dirac's initial work [14]. Dirac's idea was to extend his correspondence principle between the canonical observables q, p (i.e. the coordinate and momentum) in classical phase space and their quantum mechanical operator counterparts $\hat{q}$, $\hat{p}$, (note that we consider $\hbar = 1$ in this work)

$$(q, p) \rightarrow (\hat{q}, \hat{p}) \qquad \{q, p\} \rightarrow -\mathrm{i}[\hat{q}, \hat{p}] \tag{1}$$

to that between the classical (J, θ) and the quantum $(\hat{J}, \hat{\theta})$ formulations of the canonical action–angle observables

$$(J, \theta) \rightarrow (\hat{J}, \hat{\theta}) \qquad \{J, \theta\} \rightarrow -\mathrm{i}[\hat{J}, \hat{\theta}]. \tag{2}$$

Based on this Dirac correspondence in equation (2) it is expected that the uncertainties in the simultaneous measurement of the AA observables are related by

$$(\Delta\hat{J})(\Delta\hat{\theta}) \geqslant \tfrac{1}{2}. \tag{3}$$

Equations (2) and (3) were already known by Dirac to be approximately true for quantum fields with large intensity (i.e. J) fluctuation. It is approximate in the sense that, a manifestly Hermitian phase operator does not even exist if $\hat{J}$ spans $\mathbb{Z}^{+}$ (such as in standard QHO). Even so, if the fluctuation in the intensity is sufficiently small there is a regime in which equation (3) yields unphysical results. Particularly, if there is no fluctuation in the intensity, then equation (3) incorrectly implies an unbounded phase fluctuation despite the fact that the physical limit is $(\Delta\theta) = \pi/\sqrt{3}$ and for all other cases $(\Delta\theta) < \pi/\sqrt{3}$ should be respected.

Equation (3) is only one of a large number of inconsistencies in demanding direct analogies such as equation (2) between classical and quantum mechanics. So dramatic are the consequences that the most natural attempt to formulate even the simplest quantum system, the standard (continuous) quantum harmonic oscillator, within a canonical action–angle formalism is strictly forbidden. This can also be proven in a number of different ways. One approach can be based on the following argument.

In standard quantum mechanics the minimal representation of canonical operators is defined through the standard Fourier automorphism $\hat{\mathcal{F}}$. For instance, if we define the unitary generators of linear coordinate and momentum as $\hat{\mathcal{E}}_q = \exp(\mathrm{i}\hat{q})$ and $\hat{\mathcal{E}}_p = \exp(\mathrm{i}\hat{p})$ we have

$$\hat{\mathcal{E}}_q \xrightarrow{\hat{\mathcal{F}}} \hat{\mathcal{E}}_p \xrightarrow{\hat{\mathcal{F}}} \hat{\mathcal{E}}_q^{\dagger} \xrightarrow{\hat{\mathcal{F}}} \hat{\mathcal{E}}_p^{\dagger} \xrightarrow{\hat{\mathcal{F}}} \hat{\mathcal{E}}_q \tag{4}$$

where it is implied that $\hat{\mathcal{E}}_p = \hat{\mathcal{F}}^{-1}\hat{\mathcal{E}}_q\hat{\mathcal{F}}$ and similarly for the others. The eigenstates of coordinate $|q\rangle$ and momentum $|p\rangle$ are connected similarly as

$$|p\rangle \xrightarrow{\hat{\mathcal{F}}} |q\rangle \xrightarrow{\hat{\mathcal{F}}} |-p\rangle \xrightarrow{\hat{\mathcal{F}}} |-q\rangle \xrightarrow{\hat{\mathcal{F}}} |q\rangle \tag{5}$$

where it is implied that $|q\rangle = \hat{\mathcal{F}}|p\rangle$ and similarly for the others. We also see from above that $\hat{\mathcal{F}}^4 = 1$ and $\hat{\mathcal{F}}^2$ corresponds to the parity operator. Equations (4) and (5) are valid in a more general sense for any quantum canonical pair of observables and hence also for the generators of the AA-pair $\hat{\mathcal{E}}_J \equiv \mathrm{e}^{-\mathrm{i}\hat{J}}, \hat{\mathcal{E}}_\theta$ (i.e. we defined them as $\hat{\mathcal{E}}_J \stackrel{\hat{\mathcal{F}}}{\Longrightarrow} \hat{\mathcal{E}}_\theta$). Now, based on our correspondence with the classical case, let us naively assume that the QHO Hamiltonian can be expressed in terms of some Hermitian action operator $\hat{J}$ as $\hat{\mathcal{H}}_{QHO} = \Omega(\hat{p}^2 + \hat{q}^2)/2 = \Omega\hat{J}$, where Ω is the oscillator frequency. An application of equations (4) immediately leads to the fact that $\hat{J}$ is invariant under a Fourier automorphism and hence it has no distinct canonical partner. This result is also connected with the non-negativity of the spectrum of $\hat{J}$.

The problems with the quantum mechanical AA representation can be solved in several different ways. The one we are interested in here is based on a *canonical discretization* of the quantum system so that it permits a discrete and finite-dimensional representation of the quantum phase space [15–17] in which the canonical pairs are generated by a discrete and finite Fourier automorphism with the condition that there is a unique physical limit to the standard (continuum) formulation. More precisely, the recipe for canonical discretization we follow in this work is based on devising action and angle-related operators, defined via a discrete Fourier automorphism similar to equation (5), with their discrete eigen-spectrum defined on $\mathbb{Z}_D$. Roughly speaking, we then consider the specific asymmetric limit $D \to \infty$ which is considered in the way that the limiting spectrum of the action operator is defined in $\mathbb{Z}$ and the angle one is on the continuous circle.

There are three crucial conditions for this representation to be an appropriate basis.

(a) The existence of a unitary-discrete (hence finite-dimensional), complete and orthogonal canonical basis which is, similarly to equations (4) and (5), closed under discrete and finite Fourier automorphism.
(b) The existence (not necessarily unique) of finite-dimensional representations of operators in this basis so that it allows the discrete and finite-dimensional Hamiltonian operator to acquire a canonical partner at all finite dimensions. In other words, the Hamiltonian and the corresponding action operator should not be trivially transformed under discrete and finite Fourier transformation.
(c) The existence of a unique limit when the dimension of the discrete and finite-dimensional representations is extended to infinity in which, all such equivalent representations converge to one and the same standard continuum model.

It is now generally accepted that QHO has a key conceptual role in a generalized understanding of the AA phase space. We have examined in [16, 17] the deformed oscillator and in [18] the Kravchuk oscillator [19, 20] as examples of such discrete and finite-dimensional representations of the harmonic oscillator, satisfying all three conditions above and yielding the standard QHO in the limit as described by (c) above. Certainly it is natural that there exists a large number of such discrete and finite-dimensional representations of the QHO.

In section 2 we will give a short discussion on the von Neumann–Weyl–Heisenberg–Schwinger (vNWHS) basis [21]† as an example of a discrete, finite-dimensional, complete and orthogonal canonical operator basis in (a). Of special importance for the AA Wigner formalism particularly in mixed states is the concept of fractionally shifted spaces which are also presented in section 2. Section 3 is devoted to the discrete Wigner function and its

† The authors learned that J von Neumann had worked with what is usually known in the literature as the Weyl–Heisenberg–Schwinger basis in the early 1930s (see the third reference of [21]). It will hence be more appropriate from now on to rename our basis as the von Neumann–Weyl–Heisenberg–Schwinger basis, although we made the attribution only to Weyl, Heisenberg and Schwinger in our earlier publications. We thank the anonymous referee for this remark.

semi-discrete (AA) limit when the dimension of the discrete finite phase space is extended to infinity. The results obtained there are compared with the existing formalisms [12, 13] of the AA Wigner function.

Another point that we will be concerned with in this paper is discrete finite systems with one degree of freedom. With the help of the prime decomposition theorem of the vNWHS basis [21, 22], this implies that the dimension of the finite-dimensional canonical basis will be a prime number.

In section 4, certain discrete and finite-dimensional physical models are introduced to examine the time dependence of the AA Wigner function. Their behaviour in the continuum limit is examined and shown to yield the standard quantum mechanical results.

2. The canonical discretization

Consider a D-dimensional function space† H_D supporting orthonormal basis vectors $\{|J\rangle\} \equiv |j\rangle_{0\leqslant j\leqslant D-1}$ as the eigenbasis of some unitary operator $\hat{\mathcal{E}}_J$ with the cyclic property $|j+D\rangle \equiv |j\rangle$. A second orthonormal set $\{|\theta\rangle\} = |\theta_m\rangle_{0\leqslant m\leqslant D-1}$ exists as the eigenbasis of some other unitary operator $\hat{\mathcal{E}}_\theta$ with a similar cyclic property $|\theta_{m+D}\rangle = |\theta_m\rangle$ and

$$\begin{aligned} \hat{\mathcal{E}}_\theta|j\rangle &= |j+1\rangle \qquad & \hat{\mathcal{E}}_\theta|\theta_m\rangle &= \omega^m|\theta_m\rangle \\ \hat{\mathcal{E}}_J|j\rangle &= \omega^{-j}|j\rangle \qquad & \hat{\mathcal{E}}_J|\theta_m\rangle &= |\theta_{m+1}\rangle \end{aligned} \tag{6}$$

where $\omega = \mathrm{e}^{\mathrm{i}\gamma_0}$ and $\gamma_0 = 2\pi/D$. We then have $\hat{\mathcal{E}}_J^D = \hat{\mathcal{E}}_\theta^D = 1$ and

$$\hat{\mathcal{E}}_\theta^k \hat{\mathcal{E}}_J^\ell = \omega^{k\ell} \hat{\mathcal{E}}_J^\ell \hat{\mathcal{E}}_\theta^k. \tag{7}$$

The two bases are connected by the Fourier transformation $\hat{\mathcal{F}}$ as

$$\{|\theta\rangle\} = \hat{\mathcal{F}}\{|J\rangle\} \tag{8}$$

where the matrix elements $(\hat{\mathcal{F}})_{mj}$ are given by

$$(\hat{\mathcal{F}})_{mj} = \frac{1}{\sqrt{D}}\omega^{-mj} \tag{9}$$

and $\hat{\mathcal{F}}$ is unitary (i.e. $(\hat{\mathcal{F}}^\dagger)_{j,m} = (\hat{\mathcal{F}})^*_{m,j} = (\hat{\mathcal{F}}^{-1})_{j,m}$). The unitary operators $\hat{\mathcal{E}}_J$ and $\hat{\mathcal{E}}_\theta$ and their inverses are transformed among each other under the action of a discrete and finite Fourier transformation $\hat{\mathcal{F}}$ in a similar way to equation (4) as

$$\begin{aligned} \hat{\mathcal{F}}^{-1}\hat{\mathcal{E}}_\theta^k\hat{\mathcal{F}} &= \hat{\mathcal{E}}_J^{-k} \qquad & \hat{\mathcal{F}}^{-1}\hat{\mathcal{E}}_J^k\hat{\mathcal{F}} &= \hat{\mathcal{E}}_\theta^{k} \\ \hat{\mathcal{F}}^{-2}\hat{\mathcal{E}}_\theta^k\hat{\mathcal{F}}^2 &= \hat{\mathcal{E}}_\theta^{-k} \qquad & \hat{\mathcal{F}}^{-2}\hat{\mathcal{E}}_J^k\hat{\mathcal{F}}^2 &= \hat{\mathcal{E}}_J^{-k} \end{aligned} \tag{10}$$

where $\hat{\mathcal{F}}^4 = \mathbb{I}$ and $\hat{\mathcal{F}}^2$ is the parity operator. In the fully discrete formalism, both canonical operators $\hat{\mathcal{E}}_J$ and $\hat{\mathcal{E}}_\theta$ have identical properties and discriminative labelling such as action and angle is completely artificial. We will nevertheless make this discrimination for convenience of language by labelling J-type operators and numbers as an action and θ types as an angle. In section 3.4 a semi-discrete formulation will be introduced where such a discrimination will be a natural consequence.

We now define the fractional powers of the operators $\hat{\mathcal{E}}_J$ and $\hat{\mathcal{E}}_\theta$ by

$$\hat{\mathcal{E}}_J^\alpha = \sum_{j=-(D-1)/2}^{(D-1)/2} \omega^{-j\alpha}|j\rangle\langle j| \qquad \hat{\mathcal{E}}_\theta^\alpha = \sum_{m=-(D-1)/2}^{(D-1)/2} \omega^{m\alpha}|\theta_m\rangle\langle\theta_m| \tag{11}$$

† We will use the terminology *D-dimensional Hilbert space* for H_D, although what we mean more precisely is the space of finite and discrete functions $L_2\mathbb{Z}_D$.

where, without loss of generality, we consider that $0 \leqslant \alpha < 1$. For $\alpha \in \mathbb{Q}$ the operators $\hat{\mathcal{E}}_J^{\alpha}$ and $\hat{\mathcal{E}}_\theta^{\alpha}$ are not single valued. In particular, $\alpha = \pm\frac{1}{2}$ will be crucial for the construction of the discrete Wigner function leading to two alternative formulations (section 3.2 and remark [25]). Using equations (11) we next define the fractionally shifted eigenstates

$$\begin{aligned} \hat{\mathcal{E}}_J^{\alpha}|\theta_m\rangle &\equiv |\theta_{m+\alpha}\rangle \\ \hat{\mathcal{E}}_\theta^{\alpha}|j\rangle &\equiv |j+\alpha\rangle. \end{aligned} \tag{12}$$

The fractionally shifted sets $\{|\theta_{m+\alpha}\rangle\}$ and $\{j+\alpha\rangle\}$ are new orthonormal sets. Namely, for fixed α, they are orthonormal

$$\langle\theta_{m+\alpha}|\theta_{m'+\alpha}\rangle = \delta_{m,m'} \tag{13}$$

and they resolve the identity as

$$\mathbb{I}_\theta^{(\alpha)} = \sum_m |\theta_{m+\alpha}\rangle\langle\theta_{m+\alpha}|. \tag{14}$$

Equations (13) and (14) similarly apply to the $\{|j+\alpha\rangle\}$ basis. On the left-hand side of equation (14) the indices θ and α are actually meaningless. We nevertheless keep those indices of which use is to be made clear in the next section for fractionally shifted finite-dimensional bases.

Equations (12) imply that the fractionally shifted bases are given by

$$|j+\alpha\rangle = \frac{1}{\sqrt{D}} \sum_{m=-(D-1)/2}^{(D-1)/2} \omega^{m(j+\alpha)}|\theta_m\rangle \qquad |\theta_{m+\alpha}\rangle = \frac{1}{\sqrt{D}} \sum_{j=-(D-1)/2}^{(D-1)/2} \omega^{-j(m+\alpha)}|j\rangle. \tag{15}$$

Fractionally shifted finite-dimensional bases

Consider now a linear arbitrary operator $\hat{\mathcal{O}}$ acting on a state $|\psi\rangle$ in a finite-dimensional space H_D. The operator $\hat{\mathcal{O}}$ can be projected onto the fractionally shifted action (or angle) sector of H_D by ν (or μ) $\in [0, 1)$ by the action of the projection operator $\hat{\mathcal{P}}_\theta^{\nu}$ ($\hat{\mathcal{P}}_J^{\mu}$) which can be defined as

$$\hat{\mathcal{P}}_\theta^{(\nu)}[\hat{\mathcal{O}}] \equiv \hat{\mathcal{E}}_\theta^{-\nu}\hat{\mathcal{O}}\hat{\mathcal{E}}_\theta^{\nu} \qquad \text{and} \qquad \hat{\mathcal{P}}_J^{(\mu)}[\hat{\mathcal{O}}] \equiv \hat{\mathcal{E}}_J^{-\mu}\hat{\mathcal{O}}\hat{\mathcal{E}}_J^{\mu}. \tag{16}$$

If $\mathbb{I}_\theta^{(\mu)}$ and $\mathbb{I}_J^{(\nu)}$ describe the unit operators as defined in equation (14), the action of the projection operator on them is described by

$$\begin{aligned} \hat{\mathcal{P}}_J^{(\mu)}[\mathbb{I}_\theta]|\psi\rangle &= \mathbb{I}_\theta^{(-\mu)}|\psi\rangle = \sum_{\ell=-(D-1)/2}^{(D-1)/2} |\theta_{\ell-\mu}\rangle\langle\theta_{\ell-\mu}|\psi\rangle \\ \hat{\mathcal{P}}_\theta^{(\nu)}[\mathbb{I}_J]|\psi\rangle &= \mathbb{I}_J^{(-\nu)}|\psi\rangle = \sum_{\ell=-(D-1)/2}^{(D-1)/2} |j-\nu\rangle\langle j-\nu|\psi\rangle. \end{aligned} \tag{17}$$

It is clear in equations (16) and (17) that the product of different projections is unambiguous, namely $[\hat{\mathcal{P}}_J^{(\nu)}, \hat{\mathcal{P}}_\theta^{(\mu)}] = 0$. Furthermore, we obviously have $\hat{\mathcal{P}}_\theta^{(\nu_1)}[\hat{\mathcal{P}}_\theta^{(\nu_2)}] = \hat{\mathcal{P}}_\theta^{(\nu_1+\nu_2)}$. These relations similarly apply for $\hat{\mathcal{P}}_J^{(\mu)}$.

Using these properties of the projection operator it is easy to see that

$$\hat{\mathcal{P}}_\theta^{(\mu_1)}[\hat{\mathcal{P}}_\theta^{(\mu_2)}[\hat{\mathcal{O}}]] = \hat{\mathcal{P}}_\theta^{(\mu_1+\mu_2)}[\hat{\mathcal{O}}] = \sum_{j_1=-(D-1)/2}^{(D-1)/2} \sum_{j_2=-(D-1)/2}^{(D-1)/2} |j_1-\mu_1\rangle\langle j_2-\mu_1|(\hat{\mathcal{O}})_{j_1+\mu_2,j_2+\mu_2} \tag{18}$$

and

$$\hat{\mathcal{P}}_J^{(\nu_1)}[\hat{\mathcal{P}}_J^{(\nu_2)}[\hat{\mathcal{O}}]] = \hat{\mathcal{P}}_J^{(\nu_1+\nu_2)}[\hat{\mathcal{O}}] = \sum_{m_1=-(D-1)/2}^{(D-1)/2} \sum_{m_2=-(D-1)/2}^{(D-1)/2} |\theta_{m_1-\nu_1}\rangle\langle\theta_{m_2-\nu_1}|(\hat{\mathcal{O}})_{m_1+\nu_2,m_2+\nu_2} \tag{19}$$

hold in terms of the matrix elements of the operator $\hat{\mathcal{O}}$ in the action and angle bases. The use of equations (18) and (19) are crucial for the discrete and finite-dimensional AA Wigner function which will be demonstrated in section 3.2.

3. The Wigner function formalism

Here we will start our discussion on the continuous non-relativistic Wigner function $W_\psi(u_1, u_2) = \langle\psi|\hat{\Delta}(u_1, u_2)|\psi\rangle$ of an arbitrary quantum state $|\psi\rangle$, where u_1, u_2 are generalized canonical phase space variables and $\hat{\Delta}(u_1, u_2)$ is defined as the operator kernel, by examining the seven fundamental properties as studied by Hillery *et al* [23].

3.1. Properties of the continuous Wigner function

(a) $W_\psi(u_1, u_2)$ is real.

(b) The integral of $W_\psi(u_1, u_2)$ in one of the phase space variables yields the probability that the state $|\psi\rangle$ can be found in the eigenstate of the canonical phase space operator corresponding to the other phase space variable. Hence,

$$\int \mathrm{d}u_k W_\psi(u_1, u_2) = |\langle u_\ell|\psi\rangle|^2 \qquad k, \ell = 1, 2 \quad \text{and} \quad k \neq \ell. \tag{20}$$

(c) $W_\psi(u_1, u_2)$ is invariant under Galilean transformations

$$\big[|u_1\rangle \;\rightarrow\; |u_1 + u_0\rangle\big] \quad\Longrightarrow\quad \big[W_\psi(u_1, u_2) \;\rightarrow\; W_\psi(u_1 + u_0, u_2)\big] \tag{21}$$

and similarly for the other variable.

(d) $W_\psi(u_1, u_2)$ is covariant under phase space reflections and, independently, under time reflections. We have, under phase space reflections

$$\big[|u_k\rangle \;\rightarrow\; |-u_k\rangle\big] \Longrightarrow \quad \big[W_\psi(u_1, u_2) \;\rightarrow\; W_\psi(-u_1, -u_2)\big] \tag{22}$$

and under time reflections

$$\big[|u_1\rangle \;\rightarrow\; \langle u_1|\big] \quad\Longrightarrow\quad \big[W_\psi(u_1, u_2) \;\rightarrow\; W_\psi(-u_1, u_2)\big]. \tag{23}$$

(e) The free time evolution of the Wigner function is given by the classical equations of motion.

(f) The inner product property

$$\int \mathrm{d}u_1\, \mathrm{d}u_2\, W_\psi(u_1, u_2) W_\varphi(u_1, u_2) = \frac{1}{2\pi}|\langle\psi|\varphi\rangle|^2. \tag{24}$$

(g) If $\hat{A}$ and $\hat{B}$ are two dynamical functions of the canonical operators with $A(u_1, u_2)$, $B(u_1, u_2)$ as their Wigner–Weyl–Moyal symbols [4] given by

$$A(u_1, u_2) = \mathrm{Tr}\{\hat{A}\Delta(u_1, u_2)\} \qquad \hat{A} = \int \mathrm{d}u_1 \int \mathrm{d}u_2\, A(u_1, u_2)\hat{\Delta}(u_1, u_2). \tag{25}$$

Then,

$$\int \mathrm{d}u_1\, \mathrm{d}u_2\, A(u_1, u_2) B(u_1, u_2) = 2\pi\, \mathrm{Tr}\{\hat{A}\hat{B}\}. \tag{26}$$

Wootters [24] has initially defined a discrete analogue of the Wigner function based on three properties which partially overlap with some of those above, defined by Hillery *et al*. The first one, the projection property, is a combination of the covariance of the Wigner function under linear canonical transformations and condition (b) and (d) here. This is a much stronger condition than just (b) and (d) combined. The second property is the inner product rule which is equivalent to (f) here. The last property is the normalization which amounts to the full volume under the Wigner function being unity.

Recently [16, 17], we have shown that a discrete and finite-dimensional covariant Wigner function formalism can be established in compliance with the discrete analogues of all conditions of Hillery *et al* and the additional condition of covariant projections of Wootters. Now, we will give a brief discussion on the fully discrete Wigner function.

3.2. The properties of the fully discrete Wigner function

We consider the union of the properties defined by Hillery *et al* and by Wootters to also be fundamental ones also for the discrete Wigner function. We will demonstrate in the following that the discrete Wigner function examined below indeed satisfies these properties.

The fully discrete finite-dimensional Wigner function of a state $|\psi\rangle$ will be defined as [17, 25]

$$W_\psi(\vec{n}) = \frac{1}{D^2}\langle\psi|\hat{\Delta}(\vec{n})|\psi\rangle \tag{27}$$

where $\vec{n}$ is a compact notation for (n_1, n_2) and the normalization is such that $\sum_{\vec{n}} W_\psi(\vec{n}) = 1$. In equation (27), $\hat{\Delta}(\vec{n})$ is the operator kernel of W_ψ and the sum over $\vec{n}$ is defined on the lattice $\mathbb{Z}_D \otimes \mathbb{Z}_D$. The kernel $\hat{\Delta}(\vec{n})$ is defined by

$$\hat{\Delta}(\vec{n}) = \sum_{\vec{m}} \omega^{\vec{m}\times\vec{n}} \hat{\mathcal{S}}_{\vec{m}} \qquad \hat{\mathcal{S}}_{\vec{m}} \equiv \omega^{m_1 m_2/2} \hat{\mathcal{E}}_J^{m_1} \hat{\mathcal{E}}_\theta^{m_2} \tag{28}$$

where $\vec{m} \times \vec{n} = (m_1 n_2 - m_2 n_1)$. The properties of the kernel are given by†

$$\begin{aligned}
&\text{(a)} \qquad \mathrm{Tr}\{\hat{\Delta}(\vec{n})\} = D \\
&\text{(b)} \qquad \mathrm{Tr}\{\hat{\Delta}(\vec{n})\hat{\Delta}(\vec{m})\} = D^3 \delta_{\vec{n},\vec{m}} \\
&\text{(c)} \qquad \hat{\Delta}(\vec{n}) = \hat{\Delta}^\dagger(\vec{n}) \\
&\text{(d)} \qquad \sum_{n_1} \hat{\Delta}(\vec{n}) = D^2 |n_2\rangle\langle n_2| \\
&\text{(e)} \qquad \sum_{n_2} \hat{\Delta}(\vec{n}) = D^2 |\theta_{n_1}\rangle\langle\theta_{n_1}| \\
&\text{(f)} \qquad \sum_{\vec{n}} \hat{\Delta}(\vec{n}) = D^2
\end{aligned} \tag{29}$$

which can be proven using the properties of the $\hat{\mathcal{E}}_\theta$ and $\hat{\mathcal{E}}_J$ in section 1 or, alternatively, using those in terms of the $\hat{\mathcal{S}}_{\vec{m}}$ operators [16].

In equation (29) properties (a) and (b) imply that the kernel defines a complete (i.e. condition (a)) and orthogonal (i.e. condition (b)) operator basis. Property (c) yields the realness

† It must be stressed that the interpretation of the factor of $\frac{1}{2}$ appearing in $\omega^{m_1 m_2/2}$ in equation (28) is not unique. We adopt the standard interpretation here as the square root of $\omega^{m_1 m_2}$. Another possibility is to consider the factor of $\frac{1}{2}$ as the inverse of 2 in $\mathbb{Z}_D$ when D is a prime. (This point was suggested by Barker and it yields an alternative definition of the discrete kernel.) The roots of unity as well as the properties of the operator kernel in equation (29) are identical in both interpretations. It must nevertheless be emphasized that the limit yielding continuous or semi-discrete Wigner function is only allowed by the standard interpretation we use here.

of the Wigner function. Properties (d) and (e) are the projection properties of the kernel where the first one is the projection operator in the action and, the second one is in the angle bases which, in turn, lead to those of the Wigner function in equation (20). Property (f) yields the normalization of the Wigner function. The inner product property in equation (24) is recovered from (b) above. Property (g) in equation (26) is also a direct consequence of the completeness and the orthogonality of the kernel stated in (a) and (b) above.

Now we examine equation (26). For any dynamical operator $\hat{\mathcal{A}}$, which we assume to be an implicit function of the unitary canonical pair $\hat{\mathcal{E}}_J$ and $\hat{\mathcal{E}}_\theta$ acting on the vectors in the discrete and finite-dimensional Hilbert space, there corresponds a unique function $a(\vec{m})$ such that

$$a(\vec{m}) = \frac{1}{D}\,\mathrm{Tr}\{\hat{\mathcal{A}}\hat{\Delta}(\vec{m})\} \qquad \hat{\mathcal{A}} = \frac{1}{D^2}\sum_{\vec{m}} a(\vec{m})\hat{\Delta}(\vec{m}). \tag{30}$$

Consider two such dynamical operators $\hat{\mathcal{A}}$ and $\hat{\mathcal{B}}$. Then, using equation (30) and property (b) above we find that

$$\mathrm{Tr}\{\hat{\mathcal{A}}\hat{\mathcal{B}}\} = \frac{1}{D}\sum_{\vec{m}} a(\vec{m})b(\vec{m}) \tag{31}$$

which is the discrete version of equation (26). The Galilean boost operators are defined by their shifts in the discrete phase space variables. They are generated by the unitary canonical phase space operators $\hat{\mathcal{E}}_J^{m_1}$ and $\hat{\mathcal{E}}_\theta^{m_2}$ by

$$\hat{\mathcal{E}}_J^{m_1}|\theta_m\rangle = |\theta_{m+m_1}\rangle \qquad \text{and} \qquad \hat{\mathcal{E}}_\theta^{m_2}|j\rangle = |j+m_2\rangle \tag{32}$$

under which the kernel transforms as

$$\Delta(\vec{n}) \;\rightarrow\; \Delta(\vec{n}\,') = \begin{cases} \hat{\mathcal{E}}_J^{-m_1}\Delta(\vec{n})\hat{\mathcal{E}}_J^{m_1} = \Delta(\vec{n}+(m_1,0)) \\ \hat{\mathcal{E}}_\theta^{-m_2}\Delta(\vec{n})\hat{\mathcal{E}}_\theta^{m_2} = \Delta(\vec{n}+(0,m_2)) \end{cases} \tag{33}$$

which yields the discrete analogue of the covariance relations in equation (21). The covariance under space reflections directly follows from the covariance under the squared Fourier transformation. Using equations (10) we have

$$\Delta(\vec{n}) \;\rightarrow\; \Delta(\vec{n}\,') = \hat{\mathcal{F}}^{-2}\Delta(\vec{n})\hat{\mathcal{F}}^{2} = \Delta(-\vec{n}) \tag{34}$$

which leads to the discrete analogue of the covariance in equation (22). On the other hand, the covariance under the time reflections and the dynamical property (5) cannot be checked before we develop a discrete model for the time evolution which will be addressed in section 4.

The covariance of the kernel $\hat{\Delta}(\vec{n})$ in equation (28) under the group action of linear canonical transformations (LCT) was discussed in [16, 17]. It needs to be mentioned that the covariance under LCT can be realized as the three-parameter generalization of the Fourier covariance. The former is composed of the discrete analogues of continuous rotations, scale transformations and the hyperbolic rotations. This implies that the third condition of Wootters on the covariant projections is also satisfied by the discrete Wigner function in equation (27). We will not discuss the formal proof of the covariance under LCT and directly refer to [16, 17] as well as [26].

The Wigner function of a finite-dimensional state $|\psi\rangle$ can be computed in the *action* representation as

$$W_\psi(\vec{n}) = \frac{1}{D^2}\sum_{\vec{m}} \omega^{\vec{m}\times\vec{n}}\langle\psi|\hat{\mathcal{P}}_\theta^{(-\mu)}[\hat{\mathcal{P}}_\theta^{(\mu)}[\hat{\mathcal{S}}_{\vec{m}}]]|\psi\rangle. \tag{35}$$

It might seem utterly spurious that we added in the definition of $\Delta(\vec{n})$ the projection $\hat{\mathcal{P}}_\theta^{(-\mu)}[\hat{\mathcal{P}}_\theta^{(\mu)}[\cdot]]$ which is nothing but the identity operator. The essence of this operator will be clear when we finally construct the discrete Wigner function particularly for a mixture of pure states. We now keep the sum over $\vec{m}$ symmetric to ensure the realness of the Wigner function for non-zero fractional shifts. Equation (35) will be separated into $m_2 =$ even and $m_2 =$ odd parts as

$$W_\psi(\vec{n}) = N^{-1} \sum_{m_2=even} \omega^{-m_2 n_1} \left\{ \sum_{m_1=-(D-1)/2}^{(D-1)/2} \omega^{m_1 n_2} \langle\psi|\hat{\mathcal{P}}_\theta^{(-\mu)}[\hat{\mathcal{P}}_\theta^{(\mu)}[\hat{\mathcal{S}}_{\vec{m}}]]|\psi\rangle \right\}$$

$$+N^{-1} \sum_{m_2=odd} \omega^{-m_2 n_1} \left\{ \sum_{m_1=-(D-1)/2}^{(D-1)/2} \omega^{m_1 n_2} \langle\psi|\hat{\mathcal{P}}_\theta^{(-\mu)}[\hat{\mathcal{P}}_\theta^{(\mu)}[\hat{\mathcal{S}}_{\vec{m}}]]|\psi\rangle \right\}. \quad (36)$$

We now project the $m_2 =$ even and $m_2 =$ odd parts, respectively, on the $\mu = 0$ and $\frac{1}{2}$ sectors of the action basis. After a tedious calculation the result follows as

$$W_\psi(\vec{n}) = \frac{1}{D} \sum_{m_2=-(D=1)/2}^{(D-1)/2} \omega^{-m_2 n_1} \langle\psi|n_2 + \tfrac{1}{2}m_2\rangle\langle n_2 - \tfrac{1}{2}m_2|\psi\rangle \quad (37)$$

namely, the Wigner function, according to the m_2 summation, separates into odd and even parts as

$$W_\psi(\vec{n}) = W_\psi(\vec{n})^{(odd)} + W_\psi(\vec{n})^{(even)}. \quad (38)$$

Hence the discrete Wigner function is given by a similar form to the continuous one. Equation (37) is obtained directly by the use of the implicitly built-in fractional projections. Vaccaro [12] as well as Lukš and Peřinová [13] have also obtained equation (37). In the former an additional property of the Wigner function was introduced to cover the half-shifted spaces. In the latter, the authors have introduced by hand the half-integer action states generating the odd part of the Wigner function.

Alternatively, we could have defined $\hat{\Delta}(\vec{n})$ in equation (28) by projecting its elements onto the fractionally shifted θ basis by using the projection operator $\hat{\mathcal{P}}_J^{(-\mu)}$ as

$$\hat{\Delta}(\vec{n}) = \sum_{\vec{m}} \omega^{\vec{m}\times\vec{n}} \hat{\mathcal{P}}_J^{(-\nu)}[\hat{\mathcal{P}}_J^{(\nu)}[\hat{\mathcal{S}}_{\vec{m}}]] \quad (39)$$

where $\hat{\mathcal{P}}_J^{(\nu)}$ is the projection operator defined in equation (16). Using equation (39), $W_\psi(\vec{n})$ is then given by

$$W_\psi(\vec{n}) = \frac{1}{D^2} \sum_{\vec{m}} \omega^{\vec{m}\times\vec{n}} \langle\psi|\hat{\mathcal{P}}_J^{(-\nu)}[\hat{\mathcal{P}}_J^{(\nu)}[\hat{\mathcal{S}}_{\vec{m}}]]|\psi\rangle. \quad (40)$$

The role of the projection operator $\hat{\mathcal{P}}_J^{(\nu)}$ is now clear from the earlier discussion on $\hat{\mathcal{P}}_\theta^{(\mu)}$ in equation (36). Similarly to equation (36), and using equation (19), this time we separate the m_1 summation into even and odd parts which we project onto the $\nu = 0$ and $\frac{1}{2}$ sectors, respectively. The final result for $W_\psi(\vec{n})$ in the angle eigenbasis is

$$W_\psi(\vec{n}) = \frac{1}{D} \sum_{m_1=-(D-1)/2}^{(D-1)/2} \omega^{m_1 n_2} \langle\psi|\theta_{n_1+m_1/2}\rangle\langle\theta_{n_1-m_1/2}|\psi\rangle \quad (41)$$

where equation (38) is still valid for $m_1 =$ even and $m_1 =$ odd. The Wigner functions represented in the fractionally shifted angle and action bases in equations (37) and (41) satisfy all the fundamental properties of Hillery *et al* [23] and Wootters [24], which we now summarize as

$$
\begin{aligned}
& & & W_\psi(\vec{n}) = W_\psi^*(\vec{n}) \\
& & & \sum_{n_1=0}^{D-1} W_\psi(\vec{n}) = |\langle\psi|n_2\rangle|^2 \\
& & & \sum_{n_2=0}^{D-1} W_\psi(\vec{n}) = |\langle\psi|\theta_{n_1}\rangle|^2 \\
&|\psi\rangle \to \hat{\mathcal{E}}_\theta^{-a}|\psi\rangle & & W_\psi(\vec{n}) \to W_\psi(\vec{n}+(0,a)) \\
&|\psi\rangle \to \hat{\mathcal{E}}_J^{-b}|\psi\rangle & & W_\psi(\vec{n}) \to W_\psi(\vec{n}+(b,0)) \\
&|\psi\rangle \to \hat{\mathbb{P}}|\psi\rangle & & W_\psi(\vec{n}) \to W_\psi(-\vec{n}) \\
&|\psi\rangle \to \langle\psi| & & W_\psi(\vec{n}) \to W_\psi(-n_1, n_2) \\
& & & \sum_{\vec{n}} W_\psi(\vec{n}) W_{\psi'}(\vec{n}) = |\langle\psi|\psi'\rangle|^2 .
\end{aligned}
\tag{42}
$$

It is clear from equations (42) that the action of the operators $\hat{\mathcal{E}}_\theta^{-a}$ and $\hat{\mathcal{E}}_J^{-b}$ is equivalent to a shift in the discrete phase space coordinates n_2 and n_1, respectively, and they are the generators of the Galilean transformations. We have not included among equations (42) the dynamical property (the free time evolution described by (5) in the beginning of section 3). This property will be shown to be manifest in section 4 where a discrete Hamiltonian model for a free particle is examined.

3.2.1. Some simple examples of the discrete AA Wigner function. In this section we will expand the formal expressions (37) and (41) by considering for $|\psi\rangle$ a few examples.

(a) A finite-dimensional fractionally shifted action eigenstate: $|\psi\rangle \equiv |m+\gamma\rangle$. Inserting $|m+\gamma\rangle$ where $\gamma \in [0, 1)$ directly into equation (35) for $|\psi\rangle$ we find

$$W_{m+\gamma}(\vec{n}) = \frac{1}{D^2} \sum_{k=-(D-1)/2}^{(D-1)/2} \omega^{k(n_2-m-\gamma)}. \tag{43}$$

We plot equation (43) on the finite polar lattice in figure 1 for $m = 5$, $\gamma = 0.3$ and $D = 37$. The marginal probability distributions of the action n_2 and the angle n_1 variables are

$$P(n_2) = \sum_{n_1} W_{m+\gamma}(\vec{n}) = \frac{1}{D}\left(1 + 2\sum_{k=1}^{(D-1)/2} \cos\{k(n_2 - m - \gamma)\}\right) \qquad \gamma \in [0, 1) \tag{44}$$

and

$$\tilde{P}(n_1) = \sum_{n_2} W_{m+\gamma}(\vec{n}) = \frac{1}{D}. \tag{45}$$

In the action eigenstate, the marginal probability for the angle distribution in equation (45) is uniform in the finite range $-\frac{1}{2}(D-1) \leqslant n_1 \leqslant \frac{1}{2}(D-1)$ as expected. The action distribution in equation (44) yields a coherent δ-like distribution for $\gamma = 0$. Equations (44) and (45) can also be directly calculated from $|\psi\rangle$ consistently as

$$P(n_2) = |\langle j = n_2|\psi\rangle|^2 \qquad \tilde{P}(n_1) = |\langle\theta_{n_1}|\psi\rangle|^2. \tag{46}$$

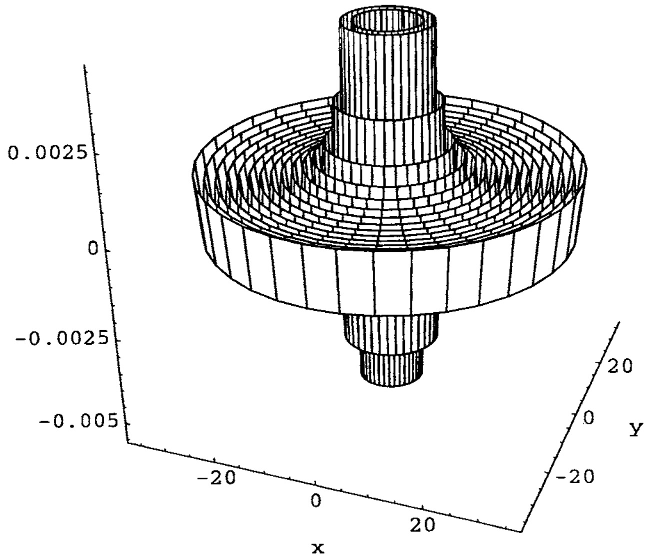

Figure 1. The discrete AA Wigner function corresponding to the fractionally shifted action eigenstate $|m+\gamma\rangle$ for $m = 5$ and $\gamma = 0.3$.

(b) A finite-dimensional fractionally shifted angle eigenstate: $|\psi\rangle \equiv |\theta_{\ell+\gamma}\rangle$ *where* $\gamma \in [0, 1)$. Starting from equation (41), a similar calculation leading to equation (43) yields

$$W_{\theta_{\ell+\gamma}}(\vec{n}) = \frac{1}{D^2} \sum_{k=-(D-1)/2}^{(D-1)/2} \omega^{-k(n_1-\ell-\gamma)}. \tag{47}$$

The marginal probability distributions for the action n_2 and the angle n_1 variables are

$$P(n_2) = \sum_{n_1} W_{\theta_{\ell+\gamma}}(\vec{n}) = \frac{1}{D} \tag{48}$$

and

$$\tilde{P}(n_1) = \sum_{n_2} W_{\theta_{\ell+\gamma}}(\vec{n}) = \frac{1}{D}\left(1 + 2\sum_{k=1}^{(D-1)/2} \cos\{-k(n_1-\ell-\gamma)\}\right) \qquad \gamma \in [0, 1). \tag{49}$$

In the angle eigenstate, the marginal distribution for the action variable is uniform, as indicated by equation (48). The angle distribution in equation (49) yields a δ-like distribution for $\gamma = 0$.

(c) The split state (or the Schrödinger cat): $|\psi\rangle_s = \frac{1}{\sqrt{2}}(|n\rangle \pm |m\rangle)$ *where* $n \neq m$. For the first two cases in equation (43) and (47) the Wigner function can also be directly obtained without the use of projection operators. The use of projection operators becomes clear particularly in mixed states, such as the split state, in which $n - m$ is an odd integer. For this case starting from equation (36) we have

$$W_{\psi_s}(\vec{n}) = \frac{1}{2D}\left[\delta_{n_2,n} + \delta_{n_2,m} \pm 2\frac{1}{D} \sum_{k=-(D-1)/2}^{(D-1)/2} \omega^{k[n_2-(n+m)/2)]} \cos\{\gamma_0 n_1(n-m)\}\right]. \tag{50}$$

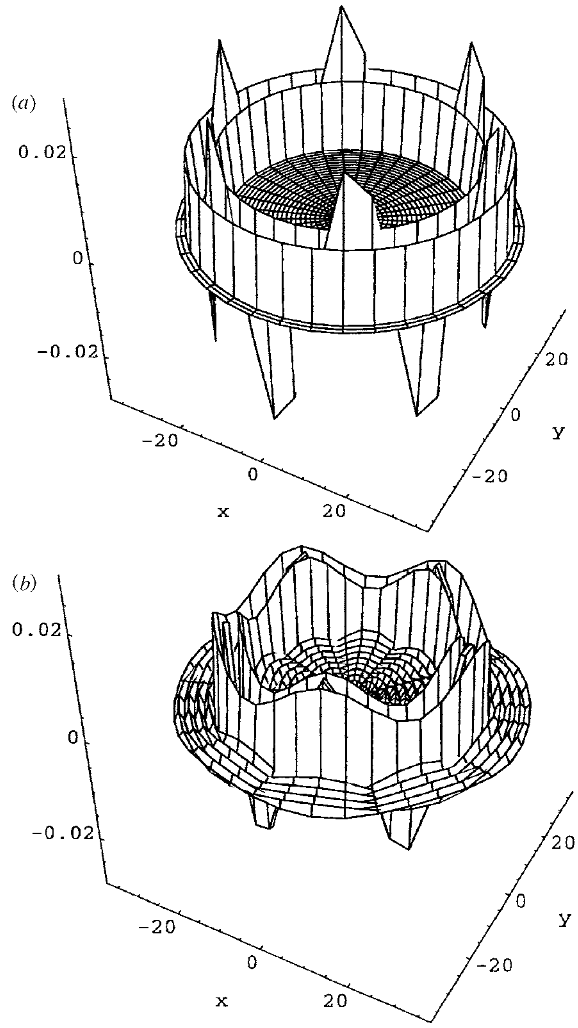

Figure 2. (*a*) The discrete AA Wigner function corresponding to the split state in action $|\psi\rangle = (|27\rangle + |33\rangle)/\sqrt{2}$ for $D = 37$. (*b*) The discrete AA Wigner function corresponding to the split state in action $|\psi\rangle = (|22\rangle + |27\rangle)/\sqrt{2}$ for $D = 37$.

The marginal probability distributions for the discrete Wigner function in the split state are then found to be

$$P(n_2) = \tfrac{1}{2}\big(\delta_{n_2,n} + \delta_{n_2,m}\big) = |\langle n_2|\psi\rangle|^2 \tag{51}$$

and

$$\tilde{P}(n_1) = \frac{1}{D}\big(1 \pm \cos\{\gamma_0 n_1(n-m)\}\big) = |\langle\theta_{n_1}|\psi\rangle|^2. \tag{52}$$

Equation (50) is depicted in figures 2(*a*) and (*b*) for two different cases in which $n+m$ is even and odd, respectively, for $D = 37$. For the even case (figure 2(*a*)) we have three radial delta functions appearing at $n_2 = n$, $n_2 = m$ and $n_2 = (n+m)/2$. The last one has an angular modulation given by the angle (n_1) dependence. For the odd case (figure 2(*b*)) the first two delta functions are still present at $n_2 = n$ and $n_2 = m$. Because we have $n+m =$ odd the third delta function at $n_2 = (n+m)/2$ in figure 2(*a*) is replaced by the $\cos\{\gamma_0[n_2 - (n+m)/2]\}$, which produces a narrow peak with a finite width and oscillations in the radial tails.

3.2.2. Calculation of the physical expectation values. The kernel $\Delta(\vec{n})$ in equation (28) provides a basis in which an arbitrary phase space operator functional $\hat{\mathcal{A}}[\hat{\mathcal{E}}_J, \hat{\mathcal{E}}_\theta]$ has a one-to-one correspondence with a unique finite and discrete function $a(\vec{n})$ through the Wigner–Weyl–Moyal correspondence as stated in equation (30). We will say that $a(\vec{n})$ is the symbol of $\hat{\mathcal{A}}$. Here we examine this correspondence explicitly for

$$\hat{\mathcal{A}}[\hat{\mathcal{E}}_J, \hat{\mathcal{E}}_\theta] = \hat{\mathcal{E}}_\theta^{\ell_1} \hat{\mathcal{E}}_J^{\ell_2}. \tag{53}$$

By directly using equation (30) and the properties in equation (29) of the kernel we find that

$$a(\vec{n}) = \frac{1}{D} \operatorname{Tr}\left\{\hat{\mathcal{E}}_\theta^{\ell_1} \hat{\mathcal{E}}_J^{\ell_2} \Delta(\vec{n})\right\} = \omega^{\ell_1 \ell_2/2} \omega^{-\vec{\ell}\times\vec{n}} \tag{54}$$

or inversely

$$\hat{\mathcal{E}}_\theta^{\ell_1} \hat{\mathcal{E}}_J^{\ell_2} = \frac{1}{D^2} \sum_{\vec{n}} \omega^{\ell_1 \ell_2/2} \omega^{-\vec{\ell}\times\vec{n}} \Delta(\vec{n}). \tag{55}$$

The expectation of $\hat{\mathcal{A}}$ in a state $|\psi\rangle$ can be computed by using the Wigner function as

$$\langle\psi|\hat{\mathcal{E}}_\theta^{\ell_1} \hat{\mathcal{E}}_J^{\ell_2}|\psi\rangle = \sum_{\vec{n}} \omega^{\ell_1 \ell_2/2} \omega^{-\vec{\ell}\times\vec{n}} W_\psi(\vec{n}). \tag{56}$$

We now show the validity of equation (56) by calculating the expectation value of the operator $\hat{\mathcal{A}}$ given by equation (53) in the split state $|\psi\rangle_s$. Using equation (50) in (56) a very short calculation yields

$${}_s\langle\psi|\hat{\mathcal{E}}_\theta^{\ell_1} \hat{\mathcal{E}}_J^{\ell_2}|\psi\rangle_s = \tfrac{1}{2}\left(\omega^{-\ell_1 n} + \omega^{-\ell_1 m}\right)\delta_{\ell_2,0} \pm \tfrac{1}{2}\left(\omega^{-\ell_1 m}\delta_{\ell_2,n-m} + \omega^{-\ell_1 n}\delta_{\ell_2,m-n}\right) \tag{57}$$

as the correct result which can be checked easily by a direct calculation of the left-hand side. A more general operator $\hat{\mathcal{A}}$ than equation (53) is an expansion in terms of $\hat{\mathcal{E}}_\theta^{\ell_1} \hat{\mathcal{E}}_J^{\ell_2}$ with some coefficients, say A_{ℓ_1,ℓ_2}. The symbol of this general operator is then $\sum_{\ell_1,\ell_2} A_{\ell_1,\ell_2} a(\vec{n})$, where $a(\vec{n})$ is given by equation (54).

3.3. The $D \to \infty$ limit and the semi-discrete AA Wigner function

Specifically in the $D \to \infty$ limit we want to conserve the discrete nature of one of the phase space coordinates (the action j) of the AA Wigner function and find the continuous limit $-\pi \leqslant \theta \leqslant \pi$ of the angle variable θ_m. This version of the Wigner function is quite different from the fully continuous (standard) version in which both coordinates are considered as continuous (as is the case with the continuous coordinate–momentum Wigner function). Hence, we prefer to keep the 'continuous limit' terminology for the fully continuous version of the Wigner function which is not to be considered in this work.

Before we can discuss this *semi-discrete* limit, an analysis of the action–angle operator basis in the limiting (infinite-dimensional) Hilbert space and the appropriate normalizations of the basis vectors should be made. In the $D \to \infty$ limit the spectra of the unitary operators $\hat{\mathcal{E}}_\theta$ and $\hat{\mathcal{E}}_J$ are arbitrarily dense and the distributions of the eigenvalues are uniform on the unit circle. Remembering that the range of the discrete variables is the symmetric range $[-(D-1)/2, (D-1)/2]$, we define the limit $D \to \infty$ as

$$\begin{aligned} &\lim_{D\to\infty} \hat{\mathcal{E}}_J^{m_1} \to \hat{\tilde{\mathcal{E}}}_J^{\gamma} \equiv e^{-i\gamma \hat{j}} && \text{where} \quad \gamma \equiv \lim_{D\to\infty} \frac{2\pi m_1}{D} \in \mathbb{R} \\ &\lim_{D\to\infty} \hat{\mathcal{E}}_\theta^{m_2} \to \hat{\tilde{\mathcal{E}}}_\theta^{m_2} && \text{where} \quad -\infty < m_2 < \infty \quad m_2 \in \mathbb{Z}. \end{aligned} \tag{58}$$

The action of $\hat{\tilde{\mathcal{E}}}_J$ and $\hat{\tilde{\mathcal{E}}}_\theta$ are defined on the infinite-dimensional continuous and everywhere differentiable Hilbert space functions. The eigenfunctions of $\hat{\tilde{\mathcal{E}}}_J$ and $\hat{\tilde{\mathcal{E}}}_\theta$ in the Hilbert space are represented by

$$\begin{aligned} &\lim_{D\to\infty} |j\rangle = |j\rangle && -\infty < j < \infty \\ &\lim_{D\to\infty} \frac{1}{\sqrt{\gamma_0}}|\theta_m\rangle \equiv |\theta\rangle && \theta = \lim_{D\to\infty} \frac{2\pi m}{D} \in \mathbb{R} \qquad \text{and} \quad -\pi \leqslant \theta < \pi \end{aligned} \tag{59}$$

with

$$\begin{aligned} &\hat{\tilde{\mathcal{E}}}_J^{\gamma}|\theta\rangle = |\theta+\gamma\rangle && \hat{\tilde{\mathcal{E}}}_J^{\gamma}|j\rangle = e^{-i\gamma j}|j\rangle \\ &\hat{\tilde{\mathcal{E}}}_\theta^{m}|j\rangle = |j+m\rangle && \hat{\tilde{\mathcal{E}}}_\theta^{m}|\theta\rangle = e^{im\theta}|\theta\rangle. \end{aligned} \tag{60}$$

Note that, we conserve the discrete index $-\infty < j < \infty$ and switch to the continuous index $-\pi \leqslant \theta \leqslant \pi$ in equations (59) and (60). The Hilbert space bases $\{|j\rangle\}_{-\infty<j<\infty}$ and $\{|\theta\rangle\}_{-\pi\leqslant\theta\leqslant\pi}$ are connected by

$$|j\rangle = \int_{-\pi}^{\pi} \frac{d\theta}{\sqrt{2\pi}} e^{i\theta j}|\theta\rangle \qquad |\theta\rangle = \frac{1}{\sqrt{2\pi}} \lim_{D\to\infty} \sum_{j=-(D-1)/2}^{(D-1)/2} e^{-i\theta j}|j\rangle \tag{61}$$

where they are appropriately normalized as $\langle j|j'\rangle = \delta_{j,j'}$ and $\langle\theta|\theta'\rangle = \delta(\theta-\theta')$. Note that the $1/\sqrt{\gamma_0}$ factor in the second equation of (59) is necessary to obtain the standard Dirac delta normalization for $\{|\theta\rangle\}$. Equations (58) and (59) imply that we now adopt the semi-discrete form

$$\lim_{D\to\infty} \hat{\mathcal{S}}_{\vec{m}} \to \hat{\tilde{\mathcal{S}}}_{\gamma,m_2} = e^{i\gamma m_2/2} \hat{\tilde{\mathcal{E}}}_J^{\gamma} \hat{\tilde{\mathcal{E}}}_\theta^{m_2} \tag{62}$$

and thus

$$\lim_{D\to\infty} \hat{\Delta}(n_1,n_2) \;\to\; \hat{\tilde{\Delta}}(\theta,n_2) = \int_{-\pi}^{\pi} \frac{d\gamma}{2\pi} \sum_{m_2=-\infty}^{\infty} e^{i(\gamma n_2 - m_2\theta)} \hat{\mathcal{P}}_\theta^{(-\mu)}[\hat{\mathcal{P}}_\theta^{(\mu)}[\hat{\tilde{\mathcal{S}}}_{\gamma,m_2}]] \tag{63}$$

for which we define the Wigner function as

$$\tilde{W}_\psi(\theta,n_2) = \frac{1}{2\pi}\langle\psi|\hat{\Delta}(\theta,n_2)|\psi\rangle \qquad -\pi \leqslant \theta \leqslant \pi \quad -\infty < n_2 < \infty. \tag{64}$$

Equations (64) and (63) yield the semi-discrete version in the limit of equation (37) as

$$\tilde{W}_\psi(\theta,n_2) = \frac{1}{2\pi} \sum_{m_2=-\infty}^{\infty} e^{-im_2\theta}\langle\psi|n_2+m_2/2\rangle\langle n_2-m_2/2|\psi\rangle. \tag{65}$$

A similar calculation for the representation of the semi-discrete version of equation (41) gives

$$\tilde{W}_\psi(\theta,n_2) = \int \frac{d\gamma}{2\pi} e^{i\gamma n_2}\langle\psi|\theta-\gamma/2\rangle\langle\theta+\gamma/2|\psi\rangle \tag{66}$$

where we appropriately normalized equations (65) and (66) as

$$\int_{-\pi}^{\pi} d\theta \sum_{n_2=-\infty}^{\infty} \tilde{W}_\psi(\theta,n_2) = 1. \tag{67}$$

3.3.1. Some examples for the semi-discrete AA Wigner function

(a) Fractionally shifted action eigenstate: $|\psi\rangle = |m+\mu\rangle$. Considering $|m+\mu\rangle$ where $\mu \in [0, 1)$ in equation (65) we have

$$\tilde{W}_{m+\mu}(\theta, n_2) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\mathrm{d}\gamma}{2\pi}\mathrm{e}^{\mathrm{i}\gamma(n_2-m-\mu)} \tag{68}$$

which reduces to the standard result when $\gamma \in \mathbb{Z}$. For instance, for $\gamma = 0$ we have

$$\tilde{W}_{m}(\theta, n_2) = \frac{1}{2\pi}\delta_{n_2,m}. \tag{69}$$

(b) Split photon state: $|\psi\rangle_s = \frac{1}{\sqrt{2}}(|n\rangle \pm |m\rangle)$ *where* n, m *are positive integers and* $n \neq m$. Using equation (63) in (64) and performing a similar calculation as that done for equation (50) we find that

$$\tilde{W}_{\psi_s}(\theta, n_2) = \frac{1}{4\pi}\left[\delta_{n_2,n} + \delta_{n_2,m} \pm 2\int_{-\pi}^{\pi}\frac{\mathrm{d}\gamma}{2\pi}\mathrm{e}^{\mathrm{i}\gamma[n_2-(n+m)/2]}\cos\{\theta(n-m)\}\right]. \tag{70}$$

The marginal probabilities for the split state are found as

$$P(n_2) = \int_{-\pi}^{\pi}\mathrm{d}\theta\, W_{\psi_s}(\theta, n_2) = \tfrac{1}{2}\left(\delta_{n_2,n} + \delta_{n_2,m}\right) \tag{71}$$

and

$$\tilde{P}(\theta) = \sum_{n_2=-\infty}^{\infty} W_{\psi_s}(\theta, n_2) = \frac{1}{2\pi}\left(1 \pm \cos\{\theta(n-m)\}\right) \tag{72}$$

which are the correct probability distributions for a split photon state.

(c) The photon coherent state. We calculate the semi-discrete Wigner function for $|\psi\rangle = |\eta\rangle$ where

$$|\eta\rangle = \mathrm{e}^{-|\eta|^2/2}\sum_{k=0}^{\infty}\frac{\eta^k}{\sqrt{k!}}|k\rangle. \tag{73}$$

Here $\eta = |\eta|\mathrm{e}^{\mathrm{i}\theta_\eta}$ is the coherent state parameter. The AA Wigner function for the coherent state is found to be

$$\tilde{W}_{\psi_c}(\theta, n_2) = \frac{\mathrm{e}^{-|\eta|^2}}{2\pi}\sum_{n,m=0}^{\infty}\frac{|\eta|^{n+m}}{\sqrt{n!m!}}\mathrm{e}^{-\mathrm{i}(\theta+\theta_\eta)(n-m)}\int_{-\pi}^{\pi}\frac{\mathrm{d}\gamma}{2\pi}\mathrm{e}^{\mathrm{i}\gamma[n_2-(n+m)/2]}. \tag{74}$$

It must be noted that n and m are positive integers and they arise from the coherent state summations in equation (73), whereas n_2 is defined over the negative and positive integers. Separating the $n = m$ term from the others in equation (74) we have

$$\tilde{W}_{\psi_c}(\theta, n_2) = \frac{\mathrm{e}^{-|\eta|^2}}{2\pi}\left[\frac{|\eta|^{n_2}}{n_2!}\Theta(n_2) + \sum_{n\neq m=0}^{\infty}\frac{|\eta|^{n+m}}{\sqrt{n!m!}}\mathrm{e}^{-\mathrm{i}(\theta+\theta_\eta)(n-m)}\int_{-\pi}^{\pi}\frac{\mathrm{d}\gamma}{2\pi}\mathrm{e}^{\mathrm{i}\gamma[n_2-(n+m)/2]}\right] \tag{75}$$

where $\Theta(n_2)$ is the step function (i.e. $\Theta(n_2) = 1$ if $0 \leqslant n_2$ and zero otherwise). Equation (75) is plotted in figure 3 for $|\eta| = 3.5$ and $\theta_\eta = 0$.

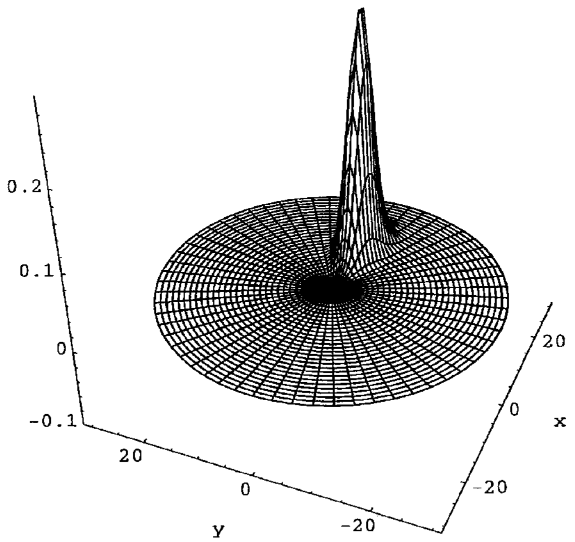

Figure 3. The AA Wigner function for the coherent state $|\eta\rangle$ in equation (73) for $|\eta| = 3.5$ and $\theta_\eta = 0$.

The marginal probabilities for the coherent state are

$$P(n_2) = \int_{-\pi}^{\pi} \mathrm{d}\theta\, W_{\psi_c}(\theta, n_2) = \frac{|\eta|^{n_2}}{n_2!}\mathrm{e}^{-|\eta|^2} \qquad 0 \leqslant n_2 \tag{76}$$

and

$$\tilde{P}(\theta) = \sum_{n_2=-\infty}^{\infty} W_{\psi_c}(\theta, n_2) = \frac{1}{2\pi}\left[1 + \mathrm{e}^{-|\eta|^2} \sum_{n\neq m=0}^{\infty} \frac{|\eta|^{n+m}}{\sqrt{n!m!}} \mathrm{e}^{-\mathrm{i}(\theta+\theta_\eta)(n-m)}\right] \tag{77}$$

where equation (76) is the Poisson number distribution for the coherent state and equation (77) is equal to the phase distribution for the coherent state which can also be directly obtained by using equations (61) and (73). We find

$$P(n_2) = |\langle n_2|\eta\rangle|^2 \qquad \text{and} \qquad \tilde{P}(\theta) = |\langle\theta|\eta\rangle|^2 \tag{78}$$

where $|\theta\rangle$ and $|\eta\rangle$ are given by equations (61) and (73), respectively. The fact that we obtained the correct Wigner function for the photon coherent state with the help of $\Theta(n_2)$ in equation (75) is actually not surprising. The only contribution to the diagonal part in equation (65) comes from the $m_2 = 0$ term in the sum decoupling the negative domain of n_2 from the positive one. Hence, the diagonal part in equation (75) is only affected by the spectrum of n_2 contributing to the wavefunction. It is guaranteed that the step function always correctly appears for those systems for which the action eigenvalues are non-negative.

4. Applications to discrete finite physical models and their continuous limits

What we imply by a physical model is a system of which dynamics is determined by a Hamiltonian operator $\hat{\mathcal{H}}_D$. Given a discrete and finite-dimensional Hamiltonian $\hat{\mathcal{H}}_D$, the time dependence of the discrete AA Wigner function is determined by

$$W_\psi(\vec{n}; t) = \langle\psi|\mathrm{e}^{\mathrm{i}\hat{\mathcal{H}}_D t}\Delta(\vec{n})\,\mathrm{e}^{-\mathrm{i}\hat{\mathcal{H}}_D t}|\psi\rangle. \tag{79}$$

Now consider the specific Hamiltonian

$$\hat{\mathcal{H}}_D = af(\hat{\mathcal{E}}_J) + \frac{b}{\kappa^2}(\hat{\mathcal{E}}_\theta^\kappa + \hat{\mathcal{E}}_\theta^{-\kappa} - 2) \tag{80}$$

where $f(\hat{\mathcal{E}}_J)$ is a real-valued operator function with $\hat{\mathcal{E}}_J, \hat{\mathcal{E}}_\theta$ as given in equation (6). The parameters a and b are real and arbitrary at the moment. There are a large number of reasons for this specific choice of Hamiltonian. In the limit $D \to \infty$ and finite κ, equation (80) has a differential operator representation in the continuous angle basis in which the standard quantum pendulum can be recovered by an appropriate choice of the function f. If one lets $\kappa \to 0$ and $D \to \infty$ independently, keeping the second-order leading term in terms of κ, one has the small-angle approximation of the quantum pendulum yielding a QHO-like system. On the other hand, some other simple integrable quantum systems can be studied for $a \neq 0, b = 0$ with an appropriate choice of the function f, including the quantum rotator on the discrete circle. The one-dimensional free particle can be recovered if the radius of the circle is extended to infinity as the square root of D (see section 4.2.6 below). Yet another physical realization of the model Hamiltonian in equation (80) is that, for $a = b$ with f as given by $f(\hat{\mathcal{E}}_J) = (\hat{\mathcal{E}}_J^\epsilon + \hat{\mathcal{E}}_J^{-\epsilon} - 2)$, where generally $\epsilon \neq \kappa$ and both real, one obtains the Harper Hamiltonian [27], which is often encountered in two-dimensional electronic systems under the influence of a constant transverse magnetic field. The Harper Hamiltonian has very interesting properties. Firstly, it is a discrete analogue of the standard QHO Hamiltonian [28]. Furthermore, it commutes with the discrete and finite Fourier transformation when $\epsilon = \kappa$. It has also been used by some workers [29] to understand the eigenspace of the fractional Fourier transform. On the other hand, following a very similar argument as discussed in the introduction, the invariance under Fourier transformation prevents Harper's Hamiltonian from having a distinct canonical partner if $\epsilon = \kappa$. As an aside we mention that, to examine the canonical partner of equation (80) in the limit $|\epsilon - \kappa| \to 0$ might be yet another interesting problem of approaching the QHO action–angle problem. Now a few examples are in order.

4.1. Some simple models including the harmonic oscillator in the limit $D \to \infty$

Here we consider a Hamiltonian represented purely in terms of one of the operators in the canonical pair $(\hat{\mathcal{E}}_J, \hat{\mathcal{E}}_\theta)$. Specifically, we will adopt $a \neq 0$ and $b = 0$ where we have

$$\hat{\mathcal{H}}_D = af(\hat{\mathcal{E}}_J). \tag{81}$$

Since we have called $\hat{\mathcal{E}}_J$ the unitary action operator from the beginning, equation (81) can be realized as an integrable quantum Hamiltonian represented purely in terms of the action invariant. The simplest standard quantum systems that can be recovered from the continuum limit of equation (81), are the QHO, the quantum rotator and the one-dimensional free particle motion. Those limits will be examined in section 4.2.

We will now examine the time dependence of the AA Wigner function related to the Hamiltonian in equation (81) and an initial state $|\psi\rangle$.

4.1.1. A pure state. Consider the eigenstates $|\psi\rangle = |j\rangle$ of equation (81). It can be seen easily that the AA-Wigner function in a pure state of action is time independent and is given by

$$W_j(\vec{n}; t) = \frac{1}{D}\delta_{n_2,j} \tag{82}$$

which is the same expression as (43).

4.1.2. A split state. We consider $|\psi\rangle_s = \frac{1}{\sqrt{2}}(|n\rangle \pm |m\rangle)$ where $n \neq m$. A similar calculation giving equation (50) yields

$$W_{\psi_s}(\vec{n}) = \frac{1}{2D}\Bigg[\delta_{n_2,n} + \delta_{n_2,m} \pm \frac{2}{D}\sum_{k=-(D-1)/2}^{(D-1)/2} \omega^{k[n_2-(n+m)/2]} \times \cos\{\gamma_0 n_1(n-m) - at[f(\omega^{-n}) - f(\omega^{-m})]\}\Bigg] \tag{83}$$

where the marginal probability distribution $P(n_2)$ is still given by the expression (51), whereas the phase (angle) probability distribution is time dependent

$$\tilde{P}(n_1; t) = \frac{1}{D}\big(1 \pm \cos\{\gamma_0 n_1(n-m) - at[f(\omega^{-n}) - f(\omega^{-m})]\}\big) \tag{84}$$

and it is properly normalized. The interesting case here is the limit $D \to \infty$. Had we used the semi-discrete Wigner function formalism instead of the fully discrete one above we would have had

$$\tilde{W}_{\psi_s}(\theta, n_2) = \frac{1}{4\pi}\Bigg[\delta_{n_2,n} + \delta_{n_2,m} \pm 2\int_{-\pi}^{\pi}\frac{\mathrm{d}\gamma}{2\pi}\mathrm{e}^{\mathrm{i}\gamma[n_2-(n+m)/2]} \times \cos\Big\{\theta(n-m) - t\lim_{D\to\infty} a[f(\omega^{-n}) - f(\omega^{-m})]\Big\}\Bigg]. \tag{85}$$

The action distribution is still given by equation (71). Whereas equation (72) has the time dependence

$$\tilde{P}(\theta; t) = \frac{1}{2\pi}\Big[1 \pm \cos\Big\{\theta(n-m) - t\lim_{D\to\infty} a[f(\omega^{-n}) - f(\omega^{-m})]\Big\}\Big] \tag{86}$$

where the $D \to \infty$ limit of the energy spectrum enters. As a specific example we choose the spectrum function f so that in the limit $D \to \infty$ we can recover the standard QHO, namely $\hat{\mathcal{H}}_\infty = \Omega\hat{J}$ with Ω being the harmonic frequency of oscillations. In particular, we have

$$f(\hat{\mathcal{E}}_J) = \frac{1}{2\mathrm{i}}(\hat{\mathcal{E}}_J - \hat{\mathcal{E}}_J^{-1}) \qquad a = \frac{\Omega}{\gamma_0} \tag{87}$$

so that $\hat{\mathcal{H}}_\infty = \lim_{D\to\infty}\hat{\mathcal{H}}_D = \lim_{D\to\infty} af(\hat{\mathcal{E}}_J) = \Omega\hat{J}$. In this QHO limit the time dependence of the phase probability distribution in equation (86) can be found by direct substitution as

$$\tilde{P}(\theta; t) = \tilde{P}(\theta(t); 0) \qquad \theta(t) = \Omega t. \tag{88}$$

This implies that the semi-discrete AA-Wigner function and thus the marginal phase probability are covariant under time evolution for the standard QHO, yielding the classical results that the action is time independent and the time dependence of the angle variable is a uniform rotation.

4.2. Finite-dimensional quantum systems on the discrete circle and their continuous limits

We now briefly examine the Hamiltonian

$$\hat{\mathcal{H}}_D = a(\hat{\mathcal{E}}_J^{\epsilon} + \hat{\mathcal{E}}_J^{-\epsilon} - 2) + \frac{b}{\kappa^2}(\hat{\mathcal{E}}_\theta^{\kappa} + \hat{\mathcal{E}}_\theta^{-\kappa} - 2) \tag{89}$$

and some of its simpler variants. The Schrödinger equation for equation (89) in the phase representation is

$$\langle\theta_m|\hat{\mathcal{H}}_D|\Phi\rangle = \left[a\Delta_\epsilon\nabla_\epsilon - 2\frac{b}{\kappa^2}(1 - \cos\kappa\theta_m)\right]\Phi(\theta_m) = E\Phi(\theta_m) \tag{90}$$

where $\Phi(\theta_m) \equiv \langle\theta_m|\Phi\rangle$ with Δ_ϵ and ∇_ϵ being the finite difference operators given by

$$\Delta_\epsilon \Phi(\theta_m) \equiv \Phi(\theta_m + \gamma_0\epsilon) - \Phi(\theta_m) \qquad \nabla_\epsilon \Phi(\theta_m) \equiv \Phi(\theta_m) - \Phi(\theta_m - \gamma_0\epsilon) \tag{91}$$

so that $\Delta_\epsilon \nabla_\epsilon$ corresponds to the second-order difference operator

$$\Delta_\epsilon \nabla_\epsilon \Phi(\theta_m) = \Phi(\theta_m + \gamma_0\epsilon) - 2\Phi(\theta_m) + \Phi(\theta_m - \gamma_0\epsilon). \tag{92}$$

If a, b are chosen appropriately so that $a\gamma_0^2\epsilon^2 \equiv \eta$ and b are finite in the limits $D \to \infty$, $\epsilon \to 0$ and $\kappa \to 0$, we can recover certain standard models in quantum mechanics. In particular, equations (90)–(92) imply that when ϵ and κ are finite in the limit $D \to \infty$ we have a quantum pendulum with a cosine-type gravitational force given by the Schrödinger equation

$$\hat{\mathcal{H}}_\infty \Phi(\theta) = -\left(\eta \frac{\partial^2}{\partial\theta^2} + 2\frac{b}{\kappa^2}(1 - \cos\kappa\theta)\right)\Phi(\theta) = E\Phi(\theta). \tag{93}$$

When $\kappa \to 0$ together with $D \to \infty$ the harmonic pendulum is obtained in the small-oscillations limit which is given by the Schrödinger equation

$$\hat{\mathcal{H}}_\infty \Phi(\theta) = -\left(\eta \frac{\partial^2}{\partial\theta^2} + b\theta^2\right)\Phi(\theta) = E\Phi(\theta). \tag{94}$$

The most general solutions of equation (89) can be called the generalized Harper functions for which no analytic solution is known. The finite-dimensional eigensolution of this model requires heavy numerical computation of which the discrete Wigner function can be examined separately.

A specific limit of the discrete quantum pendulum in equation (90) is $a \neq 0$ and $b = 0$ corresponding to zero *gravitational* interaction. This case describes the quantum rotator on the discrete circle. For finite D the discrete quantum rotator is given by the discrete Schrödinger equation

$$a\big[\Phi(\theta_m + \gamma_0\epsilon) - 2\Phi(\theta_m) + \Phi(\theta_m - \gamma_0\epsilon)\big] = E\Phi(\theta_m) \tag{95}$$

with the periodic boundary conditions $\Phi(\theta_m + 2\pi) = \Phi(\theta_{m+D}) = \Phi(\theta_m)$ satisfied. Equation (95) is an eigenproblem for the cyclic matrix

$$\hat{\mathcal{H}}_D \to a \begin{pmatrix} -2 & 1 & 0 & 0 & \dots & 0 & 1 \\ 1 & -2 & 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & -2 & 1 & \dots & 0 & 0 \\ \vdots & \dots & 0 & \dots & 0 & \ddots & 1 \\ 1 & 0 & \dots & \dots & 0 & 1 & -2 \end{pmatrix}_{D\times D} \tag{96}$$

of which the well known solution is

$$\Phi_k(\theta_m) = \frac{1}{\sqrt{D}} \mathrm{e}^{\pm \mathrm{i}\theta_m k} \tag{97}$$

and

$$E_k = -4a\sin^2(\tfrac{1}{2}\gamma_0\epsilon k) \qquad 0 \leqslant k \leqslant (D-1) \tag{98}$$

where $\eta = a\gamma_0^2\epsilon^2 = 1/(2I)$ is finite for all finite D and ϵ. Here $I = mR^2$ is defined as the moment of inertia of the rotator with mass m on the discrete circle with a fixed radius R so that the standard rotator model is recovered in the continuum limit. We now calculate the Wigner function in several initial states of this system.

4.2.1. A pure state. For a pure state we consider the *even* eigenstate

$$\psi_k(\theta_m) = \sqrt{\frac{2}{D}}\cos(\theta_m k) \qquad k \neq 0. \tag{99}$$

Inserting equation (99) in equation (27) we find

$$W_{\psi_k}(\vec{n}) = \frac{2}{D^3}\left[\sum_{m,n}\cos(\theta_m k)\cos(\theta_n k)\,\mathrm{e}^{\mathrm{i}n_2(\theta_m-\theta_n)}\sum_{\ell=-(D-1)/2}^{(D-1)/2}\mathrm{e}^{-\mathrm{i}\gamma_0\ell[n_1-(n+m)/2]}\right] \tag{100}$$

for which the marginal probabilities read

$$P(n_1) = \frac{2}{D}\cos^2(\theta_{n_1}k) \tag{101}$$

and

$$\tilde{P}(n_2) = \tfrac{1}{2}\left[\delta_{k,n_2} + \delta_{k,-n_2}\right]. \tag{102}$$

Since $k, n_2 \in \mathbb{Z}_D$ for all finite D we have $-n_2 \equiv D-n_2$. Equation (102) confirms that the state in equation (99) is a symmetric mixture of two degenerate action eigenstates. Equations (101) and (102) are, again, properly normalized. Since equation (99) describes a pure state, the corresponding Wigner function is time independent.

4.2.2. Binomial wavepacket of action. It is well known that in continuous quantum mechanics an initial state prepared as a Gaussian wavepacket spreads under the free time evolution. It is an interesting question whether there is an analogue of this problem in the discrete and finite quantum mechanics. In a discrete and finite system the natural analogue of the Gaussian wavepacket can be considered as the *binomial* wavepacket [19] (BWP). We now initially prepare our discrete quantum state in a BWP of action. This state is described in the angle representation by

$$\psi_B(\theta_m) = \frac{1}{\sqrt{D2^{(D-1)}}}\sum_{k=-(D-1)/2}^{(D-1)/2}\binom{D-1}{(D-1)/2+k}^{1/2}\mathrm{e}^{\mathrm{i}\theta_m k}. \tag{103}$$

If D is an odd prime then the dominant contribution to $\psi_B(\theta_m)$ arises in the vicinity of $k = 0$. The time dependence of an arbitrary angle state is given by

$$\psi_B(\theta_m; t) = \frac{1}{D}\sum_{\ell,k}\psi(\theta_\ell)\,\mathrm{e}^{\mathrm{i}k(\theta_\ell-\theta_m)}\mathrm{e}^{\mathrm{i}E_k t} \tag{104}$$

from which the time dependence of the wavefunction in equation (103) can be found. The Wigner function for the BWP in action then reads

$$W_{\psi_B}(\vec{n}; t) = \frac{1}{D^2}\sum_{k_1,k_2}\sum_{n,m}C^*_{k_1,n}(t)C_{k_2,m}(t)\,\omega^{n_2(n-m)}\sum_k\omega^{-k[n_1-(n+m)/2]} \tag{105}$$

where

$$C_{k_1,n}(t) = \frac{1}{\sqrt{D2^{(D-1)}}}\binom{D-1}{(D-1)/2+k_1}^{1/2}\mathrm{e}^{-\mathrm{i}\theta_n k_1}\mathrm{e}^{-\mathrm{i}E_{k_1}t} \tag{106}$$

and similarly for $C_{k_2,m}(t)$. Equation (105) is appropriately normalized. The time-dependent marginal probability distributions $P(n_1; t)$ and $\tilde{P}(n_2; t)$ are then given by

$$P(n_1; t) = \frac{1}{D}\left|\sum_{k=-(D-1)/2}^{(D-1)/2}C_{k,n_1}(t)\right|^2 = |\psi_B(\theta_{n_1}; t)|^2 \tag{107}$$

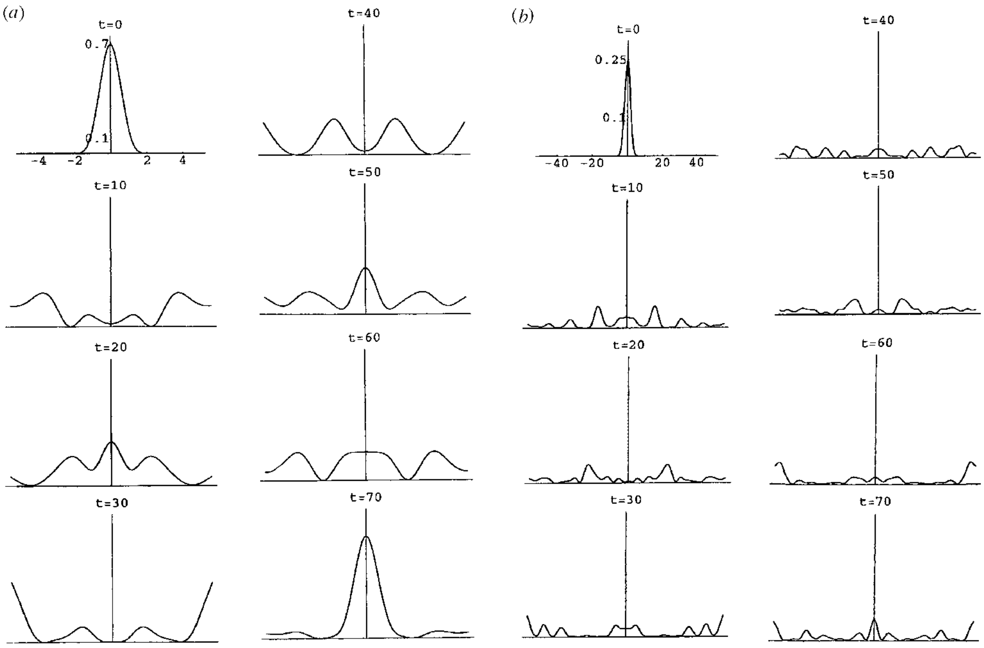

Figure 4. The time evolution of the BWP in action for the discrete quantum rotator when (*a*) $D = 11$, (*b*) $D = 101$.

and

$$\tilde{P}(n_2; t) = \frac{1}{D}\left|\sum_{k=-(D-1)/2}^{(D-1)/2}\sum_{n=-(D-1)/2}^{(D-1)/2} C_{k,n}(t)\,\omega^{-n_2 n}\right|^2 = \frac{1}{2^{(D-1)}}\binom{D-1}{(D-1)/2+n_2}. \qquad (108)$$

The second expressions on the right of equations (107) and (108) indicate that they can also be obtained directly by knowing the time dependence of the wavefunction in equation (103). The action probability distribution in equation (108) is expectedly time independent.

The smallest time scale in the time dependence of the BWP is on the order of $1/(4a)$ corresponding to the contribution of the most energetic eigenstate and the largest one is infinity corresponding to the zero-energy eigenstate. The energy eigenvalues in equation (98) are strongly incommensurate and thus the time behaviour is non-periodic. In figures 4(*a*) and (*b*) several snapshots of the angular distribution in equation (107) are presented in multiples of a fixed time interval $\Delta t < 1/(4a)$ for $D = 11$ and 101, respectively. Since the spectrum is composed of incommensurate energy eigenvalues as given by equation (98), there is no possibility for the wavefunction to recover its initial configuration. Nevertheless, for finite dimensions the number of energy eigenlevels is finite and the time behaviour is quasi-periodic. As the result, partial revivals of the wavefunction are observed and the wavefunction never *spreads* in time to a uniform distribution on the finite circle unlike in the well known continuous limit recovered in section 4.2.6 in the $D \to \infty$ limit of equation (105).

4.2.3. Binomial wavepacket of phase. The BWP in phase is

$$\tilde{\psi}_B(\theta_m) = \frac{1}{\sqrt{2^{(D-1)}}}\binom{D-1}{(D-1)/2+m}^{1/2} \qquad |m| \leqslant (D-1)/2 \qquad (109)$$

where the dominant contribution to $\tilde{\Psi}_B(\theta_m)$ comes from the vicinity of $m = 0$. The time evolution of equation (109) is calculated using equation (104). For the Wigner function we find,

$$W_{\tilde{\psi}_B}(\vec{n};t) = \frac{1}{D^2}\sum_{k_1,k_2}\sum_{n,m}\tilde{C}^*_{n,k_1}(t)\,\tilde{C}_{m,k_2}(t)\,\omega^{-n_1(k_1-k_2)}\sum_{\ell}\omega^{\ell[n_2-(k_1+k_2)/2]} \quad (110)$$

where

$$\tilde{C}_{n,k_1}(t) = \frac{1}{\sqrt{D2^{(D-1)}}}\binom{D-1}{(D-1)/2+n}^{1/2}\omega^{-nk_1}\mathrm{e}^{-\mathrm{i}E_{k_1}t}. \quad (111)$$

The time-dependent marginal probability distributions are given by

$$P(n_1;t) = \left|\sum_{m=-(D-1)/2}^{(D-1)/2}\;\sum_{k=-(D-1)/2}^{(D-1)/2}\tilde{C}_{m,k}(t)\,\omega^{-n_1k}\right|^2 = |\tilde{\psi}_B(\theta_{n_1};t)|^2 \quad (112)$$

and

$$\tilde{P}(n_2;t) = \left|\sum_{m=-(D-1)/2}^{(D-1)/2}\tilde{C}_{m,n_2}(t)\right|^2 = |\langle n_2|\tilde{\psi}_B\rangle|^2. \quad (113)$$

As expected, the action probability in equation (113) is time independent. The time dependence of the phase probability in equation (112) is depicted in figure 5 for $D = 11$.

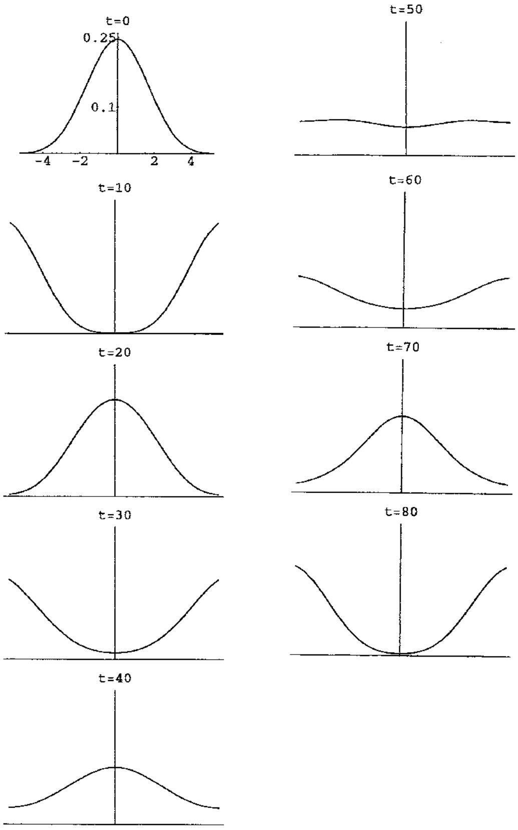

Figure 5. The time evolution of the BWP in phase for the discrete quantum rotator when $D = 11$.

4.2.4. The phase eigenstate. We now calculate the time evolution of the phase eigenstate described by $\psi(\theta_m) = \delta_{m,0}$ on the discrete circle with a large D. The time dependence of the wavefunction corresponding to the phase eigenstate can be computed using equation (104) as

$$\tilde{\psi}_B(\theta_m; t) = \frac{1}{D} \sum_{k=-(D-1)/2}^{(D-1)/2} \exp\{\mathrm{i}(k\theta_m + E_k t)\}. \tag{114}$$

Equation (114) cannot be studied analytically. We examine the time evolution of the phase probability corresponding to $D = 31\,007$ numerically for various time intervals corresponding to the multiples of the smallest time period $T_0 = 16\pi I/D^2$. The results are shown in figure 6 for $I = 1$. The wavefunction first starts to diffuse uniformly on the circle until the boundaries are reached beyond which the partial revivals and collapses are observed due to self-interference effects.

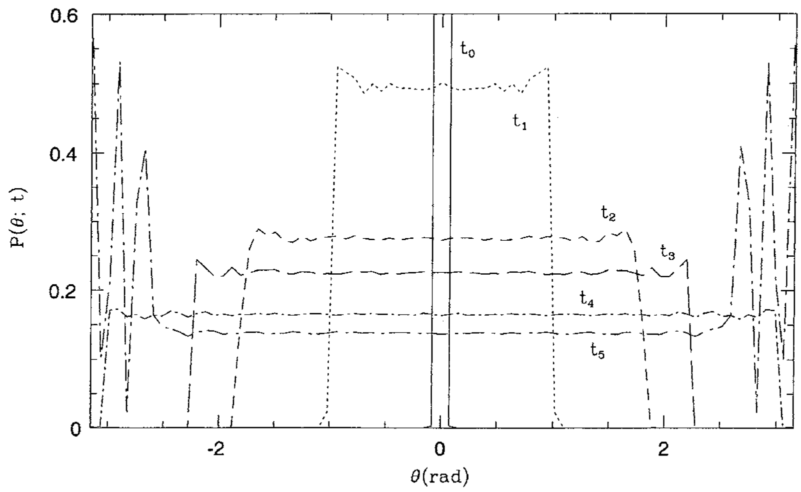

Figure 6. The time evolution on the continuous circle of a coherent phase state located at $\theta = 0$ at time $t_0 = 0$ for $t_1 = 50T_0$, $t_2 = 90T_0$, $t_3 = 110T_0$, $t_4 = 150T_0$ and $t_5 = 180T_0$. The calculation was performed for a large dimension $D = 31\,007$. Here T denotes the smallest time period where $T \sim 8\pi/D^2$.

4.2.5. Periodically kicked discrete rotator and covariant time evolution. We have seen in section 4.1 that the time evolution of the action–angle Wigner function is, generally, not covariant, i.e. $W_\Psi(\theta, n_2; t) \neq W_\Psi(\theta(t), n_2; 0)$. An exceptional case occurs in the QHO limit in equation (88). In finite phase space dimensions the violation of the covariance always occurs due to the fact that time evolution is assumed to be continuous, whereas the visited phase space points are defined on the finite-dimensional lattice. It is thus natural to ask whether a stroboscopic projection of the continuous time evolution can be covariant. We consider the finite-dimensional version of the periodically kicked rotator model as

$$\hat{\mathcal{H}}_D = af(\hat{\mathcal{E}}_J) + \hat{\mathcal{K}} \sum_{r\in\mathbb{Z}} \delta(t - rT) \tag{115}$$

where the first part is the free rotator model considered in equations (95)–(98). In the second part $\hat{\mathcal{K}}$ is some kick operator of the type described in (30) and T is the time period of the kicks.

The time evolution of an arbitrary state $|\Psi\rangle$ under equation (115) is described by [30, 31]

$$|\Psi^-(t+T)\rangle = \mathrm{e}^{-\mathrm{i}aTf(\hat{\mathcal{E}}_J)}|\Psi^+(t)\rangle \tag{116}$$

$$|\Psi^+(t)\rangle = \mathrm{e}^{-\mathrm{i}\hat{\mathcal{K}}}|\Psi^-(t)\rangle \tag{117}$$

where the superscripts $\pm$ describe the wavefunction evaluated at times infinitesimally before and after the given time instants. We now consider the case when the initial wavefunction is a phase eigenstate $|\Psi^-(0)\rangle = |\theta_m\rangle$ and consider a specific kick operator transforming a phase eigenstate into another one at the end of each time period as

$$|\theta_m^{\pm}(t+T)\rangle = |\theta_{m+m_0}^{\pm}(t)\rangle \tag{118}$$

where $m_0 \in \mathbb{Z}_D$. We find that

$$\mathrm{e}^{-\mathrm{i}\hat{\mathcal{K}}} = \hat{\mathcal{E}}_J^{-m_0}\mathrm{e}^{\mathrm{i}aTf(\hat{\mathcal{E}}_J)}. \tag{119}$$

The Wigner function for this model is identical to that of the free rotator with the same initial state except that the time evolution is now discrete in units of T, i.e. $t/T = N$ as

$$W^{\pm}(\vec{n}; N) = \frac{1}{D^3}\sum_{j,j'}\omega^{(j-j')[n_1+m+m_0(N+\binom{1}{0})]}\begin{pmatrix}\mathrm{e}^{-\mathrm{i}aT[f(\omega^{j'})-f(\omega^{j})]}\\ 1\end{pmatrix}\sum_k \omega^{k[n_2-(j+j')/2)]} \tag{120}$$

where the superscripts $\pm$ go with the upper and lower parts in the expression. The time-dependent Wigner functions before and after the time instant N are clearly covariant for any T and D

$$W^{\pm}(n_1, n_2; N) = W^{\pm}(n_1^{\pm}(N), n_2; 0) \qquad n_1^{\pm}(N) = \begin{pmatrix}n_1+m+m_0(N+1)\\ n_1+m+m_0N\end{pmatrix} \tag{121}$$

as the angle variable is only allowed to visit the designated points on the discrete circle.

Despite the explicit time dependence in equation (115), the periodically kicked model considered here is a conservative system independent from what we consider for $f(\hat{\mathcal{E}}_J)$. It can be checked directly that, for the kick operator given by equation (119) and for an arbitrary initial state, the energy of the system does not experience a discontinuous jump across a given time instant N. Hence, the model in (115) and (119) is truly the discrete-time analogue of the conservative model examined previously. We also remark that for $\hat{\mathcal{K}}$ being an arbitrary operator of the type given in equation (30) one obtains various discrete-time analogues of typical non-integrable systems. For instance, if $\hat{\mathcal{K}} = (\hat{\mathcal{E}}_\theta + \hat{\mathcal{E}}_\theta^{\dagger})$ and $f(\hat{\mathcal{E}}_J) = a(\hat{\mathcal{E}}_J + \hat{\mathcal{E}}_J^{-1} - 2)$ the discrete-time quantum nonlinear rotator [31] is obtained.

4.2.6. The continuous limit on the real line: the one-dimensional free particle and spreading of the Gaussian wavepacket. As we show below, the free particle on the real line is obtained in the limit $D \to \infty$ by letting the radius of the circle vary as the square root of the dimension D. In the limit $D \to \infty$ the binomial distribution approaches the discrete Gaussian [19]. More specifically, for $1 \ll D$ we have

$$\frac{1}{2^{(D-1)}}\binom{D-1}{(D-1)/2+k} \simeq \left(\frac{1}{\pi(D-1)/2}\right)^{1/2}\mathrm{e}^{-k^2/[(D-1)/2]}. \tag{122}$$

The time dependence of the BWP in action is given by

$$\tilde{\Psi}_B(k; t) = \lim_{D\to\infty}\left(\pi(D-1)/2\right)^{-1/4}\mathrm{e}^{-k^2/(D-1)}\mathrm{e}^{\mathrm{i}\eta k^2 t} \qquad \eta = (2I)^{-1} \tag{123}$$

of which the discrete and finite Fourier transform is the angle wavefunction corresponding to the time dependence of the initial BWP in equation (103) in the limit $D \to \infty$

$$\psi_B(\theta;t) = \frac{1}{\sqrt{2\pi}} \lim_{D\to\infty} \big(\pi(D-1)/2\big)^{-1/4} \sum_{k=-(D-1)/2}^{(D-1)/2} \mathrm{e}^{\mathrm{i}\theta k} \mathrm{e}^{-k^2/(D-1)} \mathrm{e}^{\mathrm{i}\eta k^2 t}. \tag{124}$$

In order to obtain the standard one-dimensional quantum mechanical wavefunction spread we substitute $\theta = x/R$ in equation (124), where $-\pi \leqslant \theta \leqslant \pi$, and R is the radius of the circular motion of the particle. Thus far we have considered R to be arbitrary. Consider now that $R = \sqrt{D}/\sigma$ where σ is some real and positive parameter. With this replacement in equation (124) and defining a new real variable $p = \lim_{D\to\infty} \sigma k/\sqrt{D}$ we have

$$\Phi(x;t) = \frac{1}{\sqrt{\sigma}} \left(\frac{2}{\pi}\right)^{1/4} \int_{-\infty}^{\infty} \frac{\mathrm{d}p}{\sqrt{2\pi}} \mathrm{e}^{\mathrm{i}px} \mathrm{e}^{-p^2/\sigma^2} \mathrm{e}^{\mathrm{i}tp^2/(2m)}. \tag{125}$$

Evaluating the integral we find,

$$\Phi(x;t) \equiv \frac{1}{\sqrt{R}} \Psi(\theta = x/R; t) = \left(\frac{2}{\pi}\right)^{1/4} \sqrt{\frac{\sigma}{2\Delta_t}} \exp\left\{-\frac{x^2}{4\Delta_t/\sigma^2}\right\} \tag{126}$$

where

$$\Delta_t = \left(1 - \mathrm{i}\frac{t\sigma^2}{2m}\right) \tag{127}$$

is the complex time-dependent broadening factor. Equation (126) is identical to the standard one-dimensional quantum mechanical textbook result of the free particle time evolution of the Gaussian wavepacket. To find the Wigner function, we start from equation (105). Using equation (122) and changing to the same variables used in equation (126) we expectedly find

$$W(x,p;t) = \frac{1}{\pi} \exp\left\{-2\frac{p^2}{\sigma^2}\right\} \exp\left\{-\tfrac{1}{2}\sigma^2 (x - pt/m)^2\right\} \tag{128}$$

which can also be obtained directly from equation (125) by

$$W(x,p;t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}y\, \mathrm{e}^{-\mathrm{i}yp} \Phi^*(x + y/2; t) \Phi(x - y/2; t). \tag{129}$$

Equation (128) is the well known Wigner function for the free particle in one dimension.

5. Conclusions

We have developed the theory of the discrete Wigner function for non-relativistic quantum systems with one degree of freedom and applied to a few physical examples. The conditions suggested by Hillery *et al* for the continuous Wigner function are shown to have discrete and finite-dimensional analogues that are satisfied by the discrete Wigner function.

We have also examined a few simple discrete quantum systems and derived their discrete action–angle Wigner function. In particular, the harmonic oscillator AA-Wigner function is derived and the problem with the *half-integer states* [12, 13] is resolved in a canonical and algebraic approach.

Unlike the classical scheme, the proper formulation of the quantum action–angle representation needs a canonical discretization and a proper limiting scheme from a discrete and finite to a continuous phase space. What makes the quantum formulation more difficult is the formal absence of unitary transformations from the standard $\hat{p}, \hat{q}$ representation in the action–angle one [32]. This implies that, unlike in the classical case, a nonlinear unitary canonical transformation changing the spectra of the corresponding phase space operators from that of $\hat{p}, \hat{q}$ (i.e. $(-\infty, \infty)$) to action–angle (i.e. $(\mathbb{Z}, 2\pi)$) does not exist. An explicit form for the quantum action–angle pair in terms of the $\hat{p}, \hat{q}$ is thus a highly non-trivial and interesting matter and goes beyond the standard Weyl–Wigner–Moyal formalism.

However, for all integrable continuous and bound quantum systems an action–angle representation should, in principle, exist. In particular, the Morse oscillator and the Pöschl–Teller potential may pose very interesting applications.

Apart from being good applications of the discrete Wigner function formalism examined, some of the applications themselves are interesting in their own right in finite-dimensional quantum mechanics. We introduced a discrete partner of the Gaussian wavepacket (BWP), formulated its Wigner function and examined the continuous limit. We believe that, the BWP on the discrete circle is one of a few examples of discrete finite physical systems for which the continuum limit defines a well known continuous quantum system.

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